FUNCTIONS OF LINES IN BALLISTICS*

BY

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In the theory of ballistics it is customary first of all to set up the differential equations describing the motion of a projectile under the assumptions that no wind is blowing and that the densities of the air at different altitudes are normal. If the initial velocity and the initial direction of flight of the projectile are given, it is then possible to compute the coördinates of the projectile as functions of the time t satisfying these differential equations. The path of the projectile is different, however, when a wind is blowing or when the air densities are not normal, and the form of the differential equations themselves must be changed to account for such disturbed conditions. The new differential equations involve the velocity $w(y)$ of the wind in the direction of the plane of fire at the altitude y, and the correction $\kappa(y)$ for the density at the altitude y. The range X thus becomes a function $X[w(y), \kappa(y)]$ of the functions $w(y)$, $\kappa(y)$, or in other words a function of lines, and the first differential of X in the sense of the theory of functions of lines is the first order effect of wind and density upon the range.

The range is the quantity of greatest interest to the ballistician in most problems of artillery fire. But for anti-aircraft guns one must also investigate the corrections to the three coördinates x, y, z of the projectile at the ends of successive short intervals of time t along the trajectory. It is clear that these coördinates will also be functions of lines in the sense described above for the range X.

In the following pages a method for computing the differentials of such functions of lines is described. The formulas found are the results of applications, to differential equations of the forms used in ballistics, of more general results deduced by the writer in a preceding paper.† The proofs here given, however, have been made independent as far as possible of those for the general case. The methods used have proved to be of value in the practical computations of the differential corrections hitherto deemed necessary for trajectories, and as the applications of the theory of ballistics become more

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† *Differential equations containing arbitrary functions*, these Transactions, vol. 21 (1920), p. 79.

refined the formulas will very likely be of value in the computation of other corrections also. The writer has already published accounts of the method with such mathematical details as are necessary for the practical ballistician.* The purpose of the present paper is to show the connection between the ballistic theory and the theory of functions of lines. The method given for finding the first differential of a function of lines is one which could be applied to many cases of such functions defined by differential equations.

1. The fundamental formula for differential corrections

Consider a set of differential equations of the form

$$
\begin{aligned}
x'' &= f(t, x, y, z, x', y', z'), \\
y'' &= g(t, x, y, z, x', y', z'), \qquad (1) \\
z'' &= h(t, x, y, z, x', y', z'),
\end{aligned}
$$

and having an initial solution

$$(C) \qquad x = x(t), \qquad y = y(t), \qquad z = z(t) \qquad (t_1 \leqq t \leqq t_2)$$

for which the functions $x(t)$, $y(t)$, $z(t)$ have continuous derivatives of the first and second orders on the interval $t_1 t_2$, and which passes through an initial element $(t_0, x_0, y_0, z_0, x_0', y_0', z_0')$. In a neighborhood N of the values (t, x, y, z, x', y', z') on this solution the functions f, g, h are supposed to have continuous partial derivatives up to and including those of the second order at least.

If the functions f, g, h are replaced by three others $\bar{f}, \bar{g}, \bar{h}$, having similar properties in the neighborhood N, the solutions of the equations

$$
\begin{aligned}
\bar{x}'' &= \bar{f}(t, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}'), \\
\bar{y}'' &= \bar{g}(t, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}'), \qquad (2) \\
\bar{z}'' &= \bar{h}(t, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}'),
\end{aligned}
$$

will be uniquely determined by an initial element $(t_0, x_0, y_0, z_0, \bar{x}_0', \bar{y}_0', \bar{z}_0')$, and if t_0, x_0, y_0, z_0 are kept fixed $\bar{x}$ may be regarded as a function $\bar{x}[\bar{x}_0', \bar{y}_0', \bar{z}_0', \bar{f}, \bar{g}, \bar{h}]$ with similar expressions for $\bar{y}$ and $\bar{z}$. It is natural to expect that by taking the elements of the set $[\bar{x}_0', \bar{y}_0', \bar{z}_0', \bar{f}, \bar{g}, \bar{h}]$ sufficiently near to those of the set $[x_0', y_0', z_0', f, g, h]$ belonging to the solution C the three functions

* *A method of computing differential corrections for a trajectory*, Journal of the United States Artillery, vol. 51 (1919), p. 445; *The use of adjoint systems in the problem of differential corrections for trajectories*, ibid., p. 296; *Differential corrections for anti-aircraft guns*, (not yet published).

$\bar{x}$, $\bar{y}$, $\bar{z}$ may be made to differ as little as is desired from the functions $x(t)$, $y(t)$, $z(t)$ defining that solution. That this is the case follows readily from a theorem of the first paper by the writer referred to in the introduction above,* provided that the equations ( ) are written in the normal form

$$\frac{dx}{dt} = x', \quad \frac{dy}{dt} = y', \quad \frac{dz}{dt} = z', \quad \frac{dx'}{dt} = f, \quad \frac{dy'}{dt} = g, \quad \frac{dz'}{dt} = h.$$

If Δx denotes the difference $\bar{x} - x$, with similar notations for the other variables, the quotients $\Delta x/m_1$, $\Delta x'/m_1$, and the similar ones for y and z, are all bounded as m_1 approaches zero, m_1 being the maximum of $|\Delta x'_0|$, $|\Delta y'_0|$, $|\Delta z'_0|$, $|\Delta f|$, $|\Delta g|$, $|\Delta h|$.†

By subtracting equations (1) from equations (2) a relation of the form

$$\Delta x'' = f(t, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}') - f(t, x, y, z, x', y', z') + \Delta f(t, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}')$$

is found for Δx, and similar ones hold for Δy and Δz. This equation may also be written in the form

$$(3) \quad \Delta x'' = f_x \Delta x + f_y \Delta y + f_z \Delta z + f_{x'} \Delta x' + f_{y'} \Delta y' + f_{z'} \Delta z' + \Delta f + \phi,$$

where

$$\phi = f(t, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}') - f - f_x \Delta x - f_y \Delta y - f_z \Delta z - f_{x'} \Delta x' - f_{y'} \Delta y' - f_{z'} \Delta z' + \Delta f(t, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}') - \Delta f.$$

Similar equations hold for Δy and Δz with f, ϕ replaced by g, ψ and h, χ. It is understood that arguments not indicated in f, g, h and their derivatives are always those belonging to the curve C. By applications of Taylor's formula with remainder terms of the second degree‡ to the first eight terms of the expression for ϕ, and also to the last two, it is clear that ϕ is of the second degree in Δx, Δy, Δz, $\Delta x'$, $\Delta y'$, $\Delta z'$ and the derivatives Δf_x, Δf_y, Δf_z, $\Delta f_{x'}$, $\Delta f_{y'}$, $\Delta f_{z'}$. If m_2 is the maximum of m_1 and $|\Delta f_x|$, $|\Delta f_y|$, $|\Delta f_z|$, $|\Delta f_{x'}|$, $|\Delta f_{y'}|$, $|\Delta f_{z'}|$, then ϕ/m_2^2 is bounded as m_2 approaches zero, and $\phi = m_2^2(\phi/m_2^2)$ itself may therefore be regarded as being of the second degree in m_2, as will also be the case for ψ and χ.

Equation (3) and the similar ones for $\Delta y''$ and $\Delta z''$ may be written in the normal form

* Loc. cit., p. 85.

† Loc. cit., p. 89.

‡ The formula with integral form of the remainder terms is the most satisfactory. See Jordan, *Cours d'analyse*, vol. 1, 2d edition, p. 247.

$$
(4)\qquad
\begin{aligned}
\frac{d}{dt}\Delta x &= \Delta x', \\
\frac{d}{dt}\Delta y &= \Delta y', \\
\frac{d}{dt}\Delta z &= \Delta z', \\
\frac{d}{dt}\Delta x' &= f_x\,\Delta x + f_y\,\Delta y + f_z\,\Delta z + f_{x'}\,\Delta x' + f_{y'}\,\Delta y' + f_{z'}\,\Delta z' + \Delta f + \phi, \\
\frac{d}{dt}\Delta y' &= g_x\,\Delta x + g_y\,\Delta y + g_z\,\Delta z + g_{x'}\,\Delta x' + g_{y'}\,\Delta y' + g_{z'}\,\Delta z' + \Delta g + \psi, \\
\frac{d}{dt}\Delta z' &= h_x\,\Delta x + h_y\,\Delta y + h_z\,\Delta z + h_{x'}\,\Delta x' + h_{y'}\,\Delta y' + h_{z'}\,\Delta z' + \Delta h + \chi .
\end{aligned}
$$

The system adjoint to this is by definition the system

$$
(5)\qquad
\begin{aligned}
-\lambda' &= f_x\,\rho + g_x\,\sigma + h_x\,\tau, \\
-\mu' &= f_y\,\rho + g_y\,\sigma + h_y\,\tau, \\
-\nu' &= f_z\,\rho + g_z\,\sigma + h_z\,\tau, \\
-\rho' &= \lambda + f_{x'}\,\rho + g_{x'}\,\sigma + h_{x'}\,\tau, \\
-\sigma' &= \mu + f_{y'}\,\rho + g_{y'}\,\sigma + h_{y'}\,\tau, \\
-\tau' &= \nu + f_{z'}\,\rho + g_{z'}\,\sigma + h_{z'}\,\tau,
\end{aligned}
$$

formed by using the columns of coefficients of the system (4), with signs changed, in place of the rows. If $\lambda, \mu, \nu, \rho, \sigma, \tau$ is a solution of the last system, then by multiplying the equations of the system (4) by them, respectively, and using equations (5), it follows that

$$
\frac{d}{dt}(\lambda\Delta x + \mu\Delta y + \nu\Delta z + \rho\Delta x' + \sigma\Delta y' + \tau\Delta z') = \rho\Delta f + \sigma\Delta g + \tau\Delta h + \rho\phi + \sigma\psi + \tau\chi .
$$

Furthermore by integrating this last result from t_0 to t, the relation

$$
(6)\qquad
\begin{aligned}
\lambda\Delta x + \mu\Delta y + \nu\Delta z + \rho\Delta x' + \sigma\Delta y' + \tau\Delta z' &= [\lambda\Delta x + \mu\Delta y + \nu\Delta z + \rho\Delta x' + \sigma\Delta y' + \tau\Delta z']^{t=t_0} \\
&\quad + \int_{t_0}^{t} (\rho\Delta f + \sigma\Delta g + \tau\Delta h)\,dt + \int_{t_0}^{t} (\rho\phi + \sigma\psi + \tau\chi)\,dt
\end{aligned}
$$

is found.

In his last equation the second integral is of the second degree in the maximum m_2, since ϕ, ψ, χ have this property, as explained above. The first two terms on the right may therefore be regarded as the differential of the sum on the left with respect to the differences $\Delta x_0'$, $\Delta y_0'$, $\Delta z_0'$, Δf, Δg, Δh. If t is considered as fixed for the moment, and if $(\lambda, \mu, \nu, \rho, \sigma, \tau)$ are so chosen that their initial values at the particular value t are $(1, 0, 0, 0, 0, 0)$, then the formula (6) gives an expression for Δx in which the first order terms are the first differential δx of the function $\bar{x}$ in terms of the differences $\Delta x_0'$, $\Delta y_0'$, $\Delta z_0'$, Δf, Δg, Δh. In a similar manner the differentials of $\bar{y}$ and $\bar{z}$ are determined. It is justifiable then to deduce a differential formula from (6) which takes the form

$$\begin{aligned}(7)\qquad \lambda\delta x + \mu\delta y + \nu\delta z + \rho\delta x' + \sigma\delta y' + \tau\delta z' \\ = [\rho\Delta x_0' + \sigma\Delta y_0' + \tau\Delta z_0']^{t=t_0} + \int_{t_0}^{t} (\rho\Delta f + \sigma\Delta g + \tau\Delta h)\, dt.\end{aligned}$$

It is the equation which expresses the equality of the first order terms in equation (6). The existence of the differentials of $\bar{x}$, $\bar{y}$, $\bar{z}$, and their derivatives is also an immediate consequence of the theorems of the paper referred to above.*

The last equation is fundamental for what follows. It gives the first order terms with respect to the variables $\Delta x_0'$, $\Delta y_0'$, $\Delta z_0'$, Δf, Δg, Δh of every expression of the form

$$(8)\qquad \lambda\Delta x + \mu\Delta y + \nu\Delta z + \rho\Delta x' + \sigma\Delta y' + \tau\Delta z'$$

for which $(\lambda, \mu, \nu, \rho, \sigma, \tau)$ is a solution of the adjoint system (5), and it may be called the first differential of such an expression with respect to the variables x_0', y_0', z_0' and the functions f, g, h. If the first order terms are desired at a fixed value of t for an expression of the type (8) with arbitrarily chosen coefficients, it is only necessary to insert in the equation (7) a set of solutions $\lambda, \mu, \nu, \rho, \sigma, \tau$ of equations (5) with initial values at t equal to the coefficients in the expression (8). On the other hand, if t is variable, then six linearly independent solutions of equations (5) give six independent equations of the type (7) from which the six differentials δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$ can be calculated. It is to be explained in the following sections how these considerations can be applied to calculate the differential corrections for trajectories.

2. Formulas for range corrections and deflections in ballistics

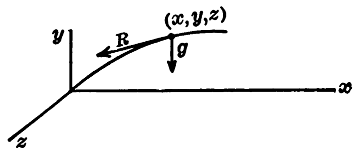
The differential equations of the trajectory of a projectile have the form

* Loc. cit., p. 89.

$$x'' = -R\frac{x'}{v} = -Fx',$$

(9) $$y'' = -R\frac{y'}{v} - g = -Fy' - g,$$

$$z'' = -R\frac{z'}{v} = -Fz',$$



where R is the retardation of the projectile in the direction of the tangent due to the resistance of the air, g is the gravitational constant, and F merely a convenient notation for R/v. The value of R used by Moulton is

$$R = vF(v, y) = v\frac{G(v)\,e^{-ay}}{C}, \qquad v = \sqrt{x'^2 + y'^2 + z'^2},$$

where $vG(v)$ is the resistance of the air to a standard projectile moving at sea level with velocity v, C is the ballistic coefficient which enables the formula to be fitted to projectiles of other types, and e^{-ay} is the altitude function which accounts for the diminution in the retardation as the altitude y increases and the density of the air diminishes. The values of $G(v)$ have been tabulated for different velocities from observations of actual firings and the table has been smoothed by means of an analytic function which agrees with the observed data with sufficient accuracy. The value of the constant $a = .000106$ for use with the meter-second system of units, is the result of meteorological observations extending over many years. It is so chosen that the function e^{-ay} represents the ratio of the density of the air at the altitude y to the density of the air at sea level under so-called normal or average conditions. These normal conditions rarely exist at an actual time of firing, and the variations from them must be taken into account as will be explained in the following paragraph.

It is presupposed that a trajectory

(10) $$x = x(t), \qquad y = y(t), \qquad z = 0 \qquad (0 \leqq t \leqq T)$$

has been computed with the help of equations (9), having given initial values $x = y = z = 0$, $x' = x_0'$, $y' = y_0'$, $z' = 0_0'$ at the time $t = 0$. The values of z will all be zero since the equations (9) have one and but one solution for the prescribed initial values, and since such a solution can be found by setting $z = 0$ in the equations and integrating the first two for x and y. The time of flight of the projectile will be denoted by T and the range by X. The actual computation may be most economically effected in a particular case by the methods of approximation devised by F. R. Moulton and his associates in the ballistic work at Washington,* the result being a table of values of the functions (10) at the ends of short intervals of time from $t = 0$ to $t = T$. The problem of the present paper is then the determination of the corrections which must be applied to the values of these functions, and in particular to the range X, in order to account for disturbances of the trajectory of various sorts, some of which will be explained below.

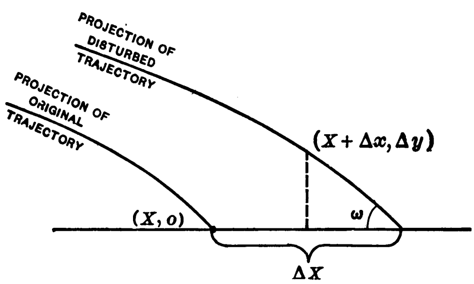
If the second members of equations (9) are denoted by f, g, h, and if they are replaced in any way by new functions

$$\bar{f} = f + \Delta f, \qquad \bar{g} = g + \Delta g, \qquad \bar{h} = h + \Delta h,$$

in order to account for disturbances of the trajectory, then the results described in Section 1 will be applicable. The linear combinations (7) of greatest interest to the ballistician are the deflection $\delta z(T)$ and the differential δX of the range X. To find a first expression for the latter one may proceed intuitively as follows. The coördinates of the projectile on the disturbed trajectory at the time T are $(X + \Delta x, \Delta y, \Delta z)$ and the value

$$\Delta X = [\Delta x + \cot \omega \Delta y]^{t=T},$$

where ω is the positive angle of fall, is approximately correct for ΔX, as one sees readily from the figure which shows the projections of the original and the disturbed trajectory on the plane of fire. The value of the differential δX is found by replacing the differences Δx and Δy by differentials in this result.



A more satisfactory method of determining δX from the standpoint of the mathematical analyst is, however, to apply the theorems on implicit functions of the paper referred to above.† The equation

$$\bar{y}[\bar{T}, \bar{x}'_0, \bar{y}'_0, \bar{z}'_0, \bar{f}, \bar{g}, \bar{h}] = 0,$$

where $\bar{y}$ is the function of Section 1 above for the special case of equations (9), has an initial solution $[T, x'_0, y'_0, 0, f, g, h]$ at the end of the trajectory (10), in a neighborhood of which the implicit function theorem is applicable. The solution $\bar{T}[\bar{x}'_0, \bar{y}'_0, \bar{z}'_0, \bar{f}, \bar{g}, \bar{h}]$ of this equation is the time of flight on the disturbed trajectory and the expression

$$X[\bar{x}'_0, \bar{y}'_0, \bar{z}'_0, \bar{f}, \bar{g}, \bar{h}] = \bar{x}[\bar{T}, \bar{x}'_0, \bar{y}'_0, \bar{z}'_0, \bar{f}, \bar{g}, \bar{h}]$$

* For a sketch of his methods see Journal of the United States Artillery, vol. 51 (1919), p. 40.

† Loc. cit., p. 90.

is the corresponding range. According to the results of the paper just cited the differential of X is

$$\delta X = \left[\delta x - \frac{x'}{y'} \delta y \right]^{t=T} = [\, \delta x + \cot \omega \delta y \,]^{t=T}, \tag{11}$$

where δx and δy are the differentials of $\bar{x}$ and $\bar{y}$ with respect to $\bar{x}_0'$, $\bar{y}_0'$, $\bar{z}_0'$, $\bar{f}$, $\bar{g}$, $\bar{h}$.

The differential equations corresponding to the system (5) for the special case of this section are readily found to be

$$\begin{aligned}
\lambda' &= 0, \\
\mu' &= -\, ax' F\rho - ay' F\sigma, \\
\nu' &= 0, \\
\rho' &= -\,\lambda + F\left(1 + x'^2 \frac{G'}{vG}\right)\rho + x'\, y'\, F \frac{G'}{vG}\sigma, \\
\sigma' &= -\,\mu + x'\, y'\, F \frac{G'}{vG}\rho + F\left(1 + y'^2 \frac{G'}{vG}\right)\sigma, \\
\tau' &= -\,\nu + F\tau.
\end{aligned} \tag{12}$$

To find δX from formulas (7) and (11) a solution $(\lambda, \mu, \nu, \rho, \sigma, \tau)$ of this system with initial values $(1, \cot \omega, 0, 0, 0, 0)$ at $t = T$ is needed. For such a solution it is clear that $\nu \equiv \tau \equiv 0$ so that the equation (7) gives

$$\delta X \doteq \rho(0)\Delta x_0' + \sigma(0)\Delta y_0' + \int_0^T (\rho\Delta f + \sigma\Delta g)\, dt. \tag{13}$$

Similarly to find $\delta z(T)$ a solution of equations (12) must be used with initial values $(0, 0, 1, 0, 0, 0)$ at $t = T$. The functions $\lambda, \mu, \rho, \sigma$ are all identically zero in this case since the first, second, fourth, and fifth of equations (12) are linear and homogeneous in $\lambda, \mu, \rho, \sigma$ alone and have only the solutions $\lambda \equiv \mu \equiv \rho \equiv \sigma \equiv 0$ with the initial values $(0, 0, 0, 0)$. Formula (7) then gives

$$\delta z(T) = \tau(0)\Delta z_0' + \int_0^T \tau \Delta h\, dt. \tag{14}$$

The function τ in the last formula can be expressed in terms of the known function $x(t)$ of the trajectory (10). For $F = -\, x''/x'$, from the first of the equations (9), and consequently the last of equations (12) gives

$$\tau' x' + \tau x'' = -\, x',$$

since now $\nu \equiv 1$. Integration of this result, and evaluation of the constant of integration by setting $t = T$, gives

$$\tau = \frac{X - x(t)}{x'(t)}. \tag{15}$$

The expressions (13) *and* (14) *are the first differentials of the range* X *and the deflection* $z(T)$ *thought of as functions of the ordinary variables* x_0', y_0', z_0' *and of the functions* f, g, h. *They give respectively the first order range correction, and the first order deflection, corresponding to a disturbance of the trajectory which can be accounted for by altering the initial conditions, or by changing the functions* f, g, h *in the second members of equations* (9) *into* $f + \Delta f$, $g + \Delta g$, $h + \Delta h$, *or both.*

3. Applications to special corrections

The formulas (13) and (14) are applicable to a very great variety of problems. Suppose, for example, that experiments had been devised which gave a new and more accurate table of values of the function $G(v)$ than the one at present used. It would not be necessary to re-compute completely the ranges corresponding under normal conditions to given elevations. For the new value of G could be inserted in equations (9) and the corrections Δf, Δg, Δh computed. The formula (13) would then give first order corrections which could be applied to revise the ranges so that they would be in accord with the new retardation law.

The most important applications of the formulas are, however, in computing the differential corrections due to variations in the initial conditions, variations from normal in the density of the air, or wind. The direction and velocity of the wind at different altitudes is determined in practice by observing the motion of a small balloon through two surveying instruments at the ends of a measured base line, and its components at the altitude y in the plane of fire and at right angles to it will be denoted hereafter by $w_x(y)$ and $w_z(y)$. The variations from normal in the density of the air at different altitudes is determined by means of airplane observations and may be accounted for in equations (9) by using an altitude function of the form $e^{-ay}[1 + \kappa(y)]$ instead of e^{-ay}.

When a wind is blowing the retardation of the projectile is no longer in the line of the tangent to the trajectory, but takes place in the direction of the motion of the projectile with respect to the air, and is determined in magnitude by the velocity V of the projectile with respect to the air. This velocity is the vector sum of the velocity of the earth with respect to the projectile and of the velocity of the air with respect to the earth, and its components and magnitude are

$$-\bar{x}' + w_x(\bar{y}), \qquad -\bar{y}', \qquad -\bar{z}' + w_z(\bar{y}),$$

$$U = \sqrt{[\bar{x}' - w_x(\bar{y})]^2 + \bar{y}'^2 + [\bar{z}' - w_z(\bar{y})]^2},$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$ represent the coördinates of the disturbed projectile. The differential equations which determine the trajectory disturbed by wind or abnormal density are now

$$
\begin{aligned}
\bar{x}'' &= -\bar{F}[\bar{x}' - w_x(\bar{y})], \\
\bar{y}'' &= -\bar{F}\bar{y}' - g, \qquad (16)\\
\bar{z}'' &= -\bar{F}[\bar{z}' - w_z(\bar{y})],
\end{aligned}
$$

where

$$
\bar{F} = \frac{1}{C} G(V) e^{-a\bar{y}}[1 + \kappa(\bar{y})]. \qquad (17)
$$

The solutions of these equations are functions $\bar{x}$, $\bar{y}$, $\bar{z}$ of the arguments t, x_0', y_0', z_0', $w_x(y)$, $w_z(y)$, $\kappa(y)$, and the first order deflection $\delta z(T)$ and range correction δX are given by formulas (14) and (13) when Δf, Δg, Δh are replaced by the first order terms of their expansions in powers of $w_x(y)$, $w_z(y)$, $\kappa(y)$ which are described below.

The second members of equations (16) are the functions previously designated by $\bar{f}$, $\bar{g}$, $\bar{h}$, and the differences $\Delta f = \bar{f} - f$, Δg, Δh can be readily calculated. In the formula (7) their arguments are the elements t, x, y, z, x', y', z' of the known solution C, and in the formulas (13) and (14) it is to be remembered that they must similarly be the analogous elements of the solution (10) already computed. It follows then that to first order terms in $w_x(y)$, $w_z(y)$, $\kappa(y)$

$$
\begin{aligned}
\Delta f &= F\left(1 + x'^2 \frac{G'}{vG}\right) w_x(y) - x' F \kappa(y) + \cdots, \\
\Delta g &= x' y' F \frac{G'}{vG} w_x(y) - y' F \kappa(y) + \cdots, \\
\Delta h &= F w_z(y) + \cdots.
\end{aligned}
$$

When these values are substituted in (13) and (14), and use is made of the equations (12) and (15), it is found that

$$
\delta X = \rho(0)\Delta x_0' + \sigma(0)\Delta y_0' + \int_0^T (\rho' + 1) w_x(y)\, dt + \frac{1}{a}\int_0^T \mu' \kappa(y)\, dt, \qquad (18)
$$

$$
\delta z(T) = \frac{X}{x_0'} \Delta z_0' + \int_0^T (\tau' + 1) w_z(y)\, dt. \qquad (19)
$$

The expressions (18) *and* (19) *are the first differentials of the range X and the deflection $z(T)$ thought of as functions of the ordinary variables x_0', y_0', z_0' and of the wind- and density-functions $w_x(y)$, $w_z(y)$, $\kappa(y)$. They give, respectively, the first order range correction and the deflection due* (1) *to a variation in the initial velocity with projections $\Delta x_0'$, $\Delta y_0'$, $\Delta z_0'$ on the coördinate axes,* (2) *to a wind at the altitude y with components $w_x(y)$, $w_z(y)$ in the plane of fire and at right angles to it, and* (3) *to a variation from normal in the density ratio of the air which at altitude y is equal to the fraction $\kappa(y)$ of the normal* *density ratio* e^{-av}. *The functions* $\lambda, \mu, \rho, \sigma$ *occurring in these equations are solutions of the first, second, fourth, and fifth of equations* (12) *with initial values* $(1, \cot\omega, 0, 0)$ *at* $t = T$, ω *being the positive angle of fall. The function* τ *is given by the equation* (15).

In constructing a range table it is sometimes desired to compute the first order correction to the range X due to the use of a slightly different value $C + \Delta C$ of the ballistic coefficient. The formula (18) will give this correction also, if the function $\kappa(y)$ is everywhere replaced by $\kappa(y) - \Delta C/C$. For the equations (16) still hold provided that the expression (17) for $\bar{F}$ is multiplied by the factor

$$\frac{1}{1 + \dfrac{\Delta C}{C}} = 1 - \frac{\Delta C}{C} + \cdots,$$

which gives, to first order terms, the value

$$\bar{F} = \frac{1}{C} G(V) e^{-av} \left[1 + \kappa(y) - \frac{\Delta C}{C} + \cdots \right].$$

This is the same as before except that $\kappa(y)$ is replaced by $\kappa(y) - \Delta C/C$.

The versatility of the formulas (13) and (14) of Section 2 is still further illustrated by the problem of determining the effect of the rotation of the earth upon a trajectory. Roever has set up the differential equations of motion in the form

$$\begin{aligned} \bar{x}'' &= -\bar{F}\bar{x}' + 2\omega(\bar{y}' \cos\phi \sin\alpha - \bar{z}' \sin\phi), \\ \bar{y}'' &= -\bar{F}\bar{y}' - g - 2\omega \cos\phi (\bar{z}' \cos\alpha + \bar{x}' \sin\alpha), \qquad (20) \\ \bar{z}'' &= -\bar{F}\bar{z}' + 2\omega(\bar{x}' \sin\phi + \bar{y}' \cos\phi \cos\alpha). \end{aligned}$$

In these formulas ϕ is the latitude of the gun position, α is the azimuth of the plane of fire measured clockwise from the south, $\omega = .0000729$ is the angular velocity of the earth's rotation in radians per second, and $\bar{F}$ is the value (17) except that now $w_x(y) = w_z(y) = \kappa(y) = 0$, since only the effect of the rotation of the earth apart from other disturbances is to be considered. The effects of all the disturbances could be evaluated simultaneously if desired but the result would be merely to add the terms about to be deduced to the expressions (18) and (19). In the equations (20) powers of ω higher than the first have been neglected.

The second member of the first of equations (20) is now the function $\bar{f}(t, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}')$, and the value of Δf in the formula (13) is the difference $\bar{f} - f$ formed for the arguments t, x, y, z, x', y', z' of the initial trajectory (10), with similar values for Δg and Δh. It follows readily that

$$\Delta f = 2\omega y' \cos\phi \sin\alpha,$$

$$\Delta g = -2\omega x' \cos\phi \sin\alpha,$$

$$\Delta h = 2\omega(x' \sin\phi + y' \cos\phi \cos\alpha),$$

and the formulas (13) and (14) give

$$\delta X = 2\omega \cos\phi \sin\alpha \int_0^T (\rho y' - \sigma x')\,dt, \tag{21}$$

$$\delta z(T) = 2\omega \int_0^T \tau(x' \sin\phi + y' \cos\phi \cos\alpha)\,dt. \tag{22}$$

These are the first differentials of the range X and the deflection $z(T)$ considered as functions of the angular velocity ω of the rotation of the earth. They give the first order effects of the rotation of the earth upon X and $z(T)$.

The advantage, from the standpoint of the computer, of the formulas for the corrections derived in this section lies in the fact that only two integrations of systems of differential equations by methods of approximation have to be made in order to get all of the corrections. The first integration is of the system (9) in order to find the original trajectory (10), and the second is the integration of the first, second, fourth, and fifth of equations (12) for the functions $(\lambda, \mu, \rho, \sigma)$ with initial values $(1, \cot\omega, 0, 0)$ at $t = T$. When this last set of functions has been determined the formulas (18), (19), (21), (22) give the values of the desired corrections by elementary operations of arithmetic, except in the case of the rotation of the earth for which two quadratures by methods of approximation are required. The details of the computation are described in the papers mentioned above* and need not be reproduced here. The methods in use when these results were obtained involved a separate approximate integration of a system of differential equations for each type of correction, and this multiplicity of integrations is now replaced by two, the amount of computational work required being diminished by about three fourths.

4. The differentials of the coördinates x, y, z for anti-aircraft corrections

The details of the computation of anti-aircraft corrections have been described by the writer in another paper.† Theoretically the steps used to obtain them are the following.

A fundamental system of four solutions $(\lambda_i, \mu_i, \rho_i, \sigma_i)$ $(i = 1, \cdots, 4)$ of the first, second, fourth, and fifth of equations (12) gives when substituted in equation (7) four linear relations

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† Submitted for publication to the Journal of the United States Artillery.

(23) $$\lambda_i\,\delta x + \mu_i\,\delta y + \rho_i\,\delta x' + \sigma_i\,\delta y' = m_i \qquad (i = 1, \cdots, 4).$$

If the fundamental system chosen is the one whose elements reduce to the identity matrix at $t = 0$, the four values m_i from the second members of equation (7) are

$$m_1 = \int_0^{t} (\rho_1\,\Delta f + \sigma_1\,\Delta g)\,dt, \qquad m_2 = \int_0^t (\rho_2\,\Delta f + \sigma_2\,\Delta g)\,dt,$$

$$m_3 = \Delta x_0' + \int_0^t (\rho_3\,\Delta f + \sigma_3\,\Delta g)\,dt,$$

$$m_4 = \Delta y_0' + \int_0^t (\rho_4\,\Delta f + \sigma_4\,\Delta g)\,dt.$$

The differentials of x and y are then found by solving the four linear equations (23) for δx, δy, $\delta x'$, $\delta y'$, and these differentials are the first order corrections to x and y required at the time t on the trajectory.

The effectiveness of this method of determining the differential corrections from the standpoint of the computer depends very much upon the plan of the computations and the arrangement of the work. Such a plan has already been described by the writer in the paper last referred to. It turns out that the fundamental system of solutions $(\lambda_i, \mu_i, \rho_i, \sigma_i)$ $(i = 1, \cdots, 4)$, and the cofactors of the determinant of the coefficients in the first members of equations (23), can be determined by three approximate integrations plus elementary arithmetical operations. The system of differential equations actually integrated is the adjoint of the first, second, fourth, and fifth equations of the system (12) which has the form

(24) $$\begin{aligned} A' &= C, \\ B' &= D, \\ C' &= ax'FB - F\left(1 + x'^2\frac{G'}{vG}\right)C - x'y'F\frac{G'}{vG}D, \\ D' &= ay'FB - x'y'F\frac{G'}{vG}C - F\left(1 + y'^2\frac{G'}{vG}\right)D. \end{aligned}$$

If (A, B, C, D) is a solution of this system, and $(\lambda, \mu, \rho, \sigma)$ a solution of the original one, the relation

$$A\lambda + B\mu + C\rho + D\sigma = \text{const.},$$

is always true, and it follows readily that four solutions (A_i, B_i, C_i, D_i) $(i = 1, \cdots, 4)$ of the system (24), with initial values at $t = 0$ forming the identity matrix, will have elements which are the cofactors of the corresponding elements of the determinant of the solutions $(\lambda_i, \mu_i, \rho_i, \sigma_i)$ $(i = 1, \cdots, 4)$

of the original system. The matrix of solutions of the system (24) has the form

$$
\begin{vmatrix}
1 & 0 & 0 & 0 \\
A_2 & B_2 & C_2 & D_2 \\
A_3 & B_3 & C_3 & D_3 \\
A_4 & B_4 & C_4 & D_4
\end{vmatrix} \tag{25}
$$

and it is clear from the form of the equations themselves that when the elements B, C, D of one of the last three rows have been determined by integration of the last three differential equations, the corresponding A can be determined by a quadrature. The elements of the solutions $(\lambda_i, \mu_i, \rho_i, \sigma_i)$ $(i = 1, \cdots, 4)$ are the co-factors of the matrix (25) and they can be calculated by elementary operations of arithmetic.

The differential of z is found by using in equation (7) two solutions (ν, τ) of the third and sixth of equations (12) with initial values $(\nu_1, \tau_1) = (1, 0)$, $(\nu_2, \tau_2) = (0, 1)$ at $t = 0$. These solutions are, respectively,

$$
\nu_1 \equiv 1, \qquad \tau_1 \equiv -\frac{x}{x'},
$$

$$
\nu_2 \equiv 0, \qquad \tau_2 \equiv \frac{x_0'}{x'},
$$

as may readily be established by elementary integrations similar to those used in deducing formula (15). Equation (7) then gives

$$
\delta z - \frac{x}{x'}\,\delta z' = -\int_0^t \frac{x}{x'}\,\Delta h dt,
$$

$$
\frac{x_0'}{x'}\,\delta z' = \Delta z_0' + \int_0^t \frac{x_0'}{x'}\,\Delta h dt,
$$

which determine the differentials δz and $\delta z'$.